Non-Invertible Symmetries

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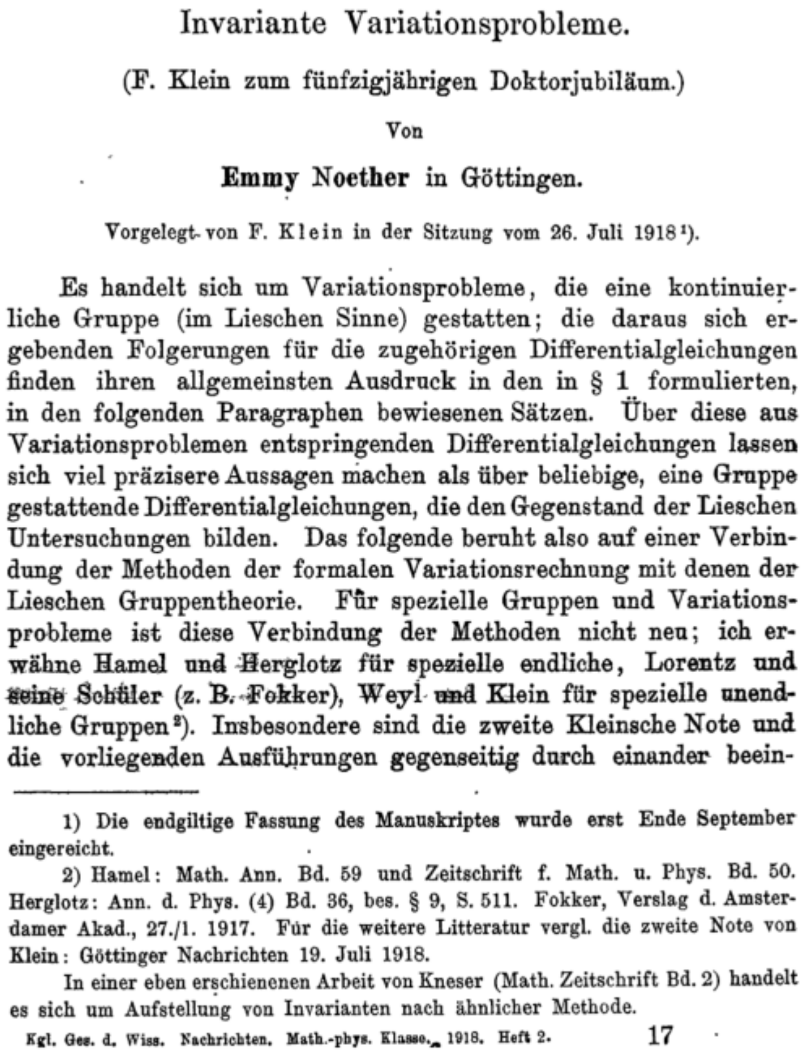
Symmetries from Topological Operators

2022: The symmetries of a QFT is generated by the set of topological defects in the theory.

This is a long way from Noether's 1918 continuous "Lieschen" type symmetries, though the core idea is the same:



Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

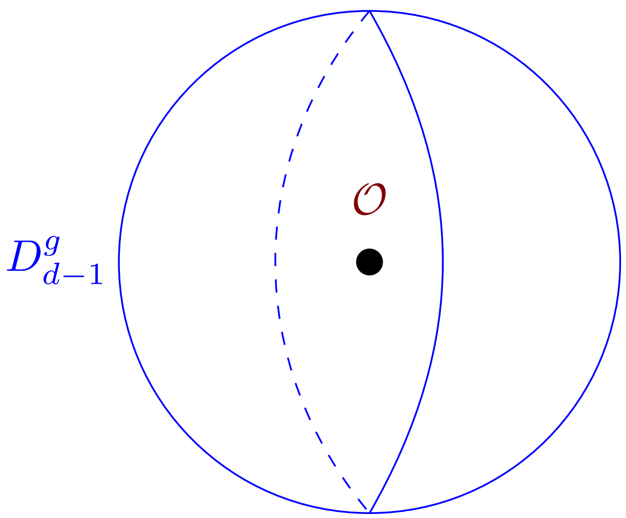
Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

Kgl. Ges. d. Wiss. Nachrichten. Math.-phys. Klasse. 1918. Heft 2. 17



Generalized Global Symmetries

Recent explosion of symmetries:

1. Higher-form symmetries $\Gamma^{(p)}$:
   charged objects are p-dimensional defects, charge measured by topological operators $D^g_{d-(p+1)}$.
2. Higher-group symmetries:
   p-form symmetries might not form product groups
3. Non-invertible symmetries:
   relax group law $\Rightarrow$ fusion algebra
4. Higher-categorical symmetries:
   topological operators of dimensions $0, \cdots, d-1$, with non-invertible fusion

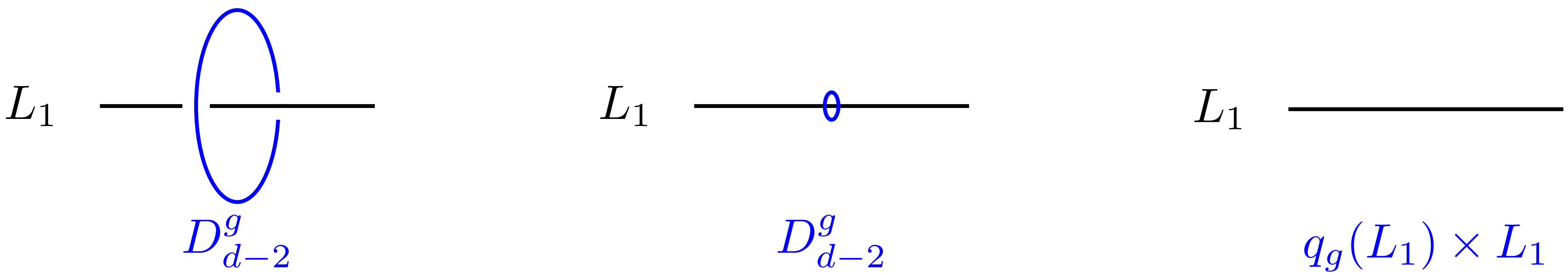
1. Higher-form symmetries $\Gamma^{(p)}$

p-dimensional charged defects, whose charge is measured by topological operators $D^g_{q=d-(p+1)}$, $g \in \Gamma^{(p)}$ [Gaiotto, Kapustin, Seiberg, Willett, 2014]

$$D^g_q \otimes D^h_q = D^{gh}_q \,, \qquad g, h \in \Gamma^{(p)}$$

p-dim extended operators links with these, and can be charged under $\Gamma^{(p)}$:

E.g. $p = 1$



The topological operators D_{d-2} are the Gukov-Witten operators.

Higher-Form Symmetries

- Background field: $B_{p+1} \in H^{p+1}(M, \Gamma^{(p)})$
- Gauging: summing over all such B.
- Intuitive way to think about these: $\Gamma^{(1)}$ is the set of line operators modulo junctions by local operators.

Example:

pure G (simply-connected) Yang Mills theory. The genuine line operators are Wilson lines, and local operators are in the adjoint, so

$$\Gamma^{(1)} = Z_G = \text{Center}(G)$$

Screening by matter, depends on charge of reps under center.

Physics: $\Gamma^{(1)}$ provides an order parameter for confinement.

2. Higher-Group Symmetries

Higher-form symmetries might not form product groups, e.g. $\Gamma^{(1)} \times \Gamma^{(0)}$, but a group extension. [Sharpe][Tachikawa][Benini, Cordova, Hsin....]

Warmup: 0-form symmetries as extension groups

Famously there are two finite groups of order 4:
[http://www.youtube.com/watch?v=BipvGD-LCjU]

$$\text{Klein} = \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad \text{and} \qquad \mathbb{Z}_4 \,.$$

$\mathbb{Z}_4$ can be thought of as a non-trivial extension

$$1 \to \mathbb{Z}_2^A \to \mathbb{Z}_4 \to \mathbb{Z}_2^B \to 1$$

These are characterized by the non-trivial element in the group cohomology $H^2(\mathbb{Z}_2^B, \mathbb{Z}_2^A) = \mathbb{Z}_2$.

2-Group Symmetries

Let $\mathbb{Z}_2^A$ be $\Gamma^{(1)}$, and $\mathbb{Z}_2^B$ part of a the 0-form flavor symmetry F. Two intuitive ways to think about 2-groups:

- Line operators (charged under $\Gamma^{(1)}$) can be screened by local operators which carry charge under the flavor symmetry
  $\Rightarrow$ introduces an extension of $\Gamma^{(1)}$ by flavor data.
- Background fields: B_2 background for $\Gamma^{(1)}$ and B_1 flavor bundle:

$$\delta B_2 = B_1^* \Theta$$

where $\Theta \in H^2(BF, \Gamma^{(1)})$, and $B_1 : M_d \to BF$.

Examples: at least as frequent as mixed 't Hooft anomalies.

4d Spin$(4N+2)$ + **V** matter [Lee, Ohmori, Tachikawa]
5d $SU(2)_0$ SCFT [Apruzzi, Bhardwaj, Oh, SSN]
6d SCFTs classified all theories with 2-groups [Apruzzi, Bhardwaj, Gould, SSN]

3. & 4. Non-Invertible and Higher-Categorical Symmetries

So far: all topological defects had fusions obeying group multiplication (in particular there was an inverse to each generator).

- Non-invertible symmetries:
  relax group law $\Rightarrow$ fusion algebra

$$D_p^i \otimes D_p^j = \bigoplus_k N_k^{ij} D_p^k$$

  This is very well developed in 2d and to some extent 3d (cond-mat), but unchartered until recently in $d > 3$.

- Higher-categorical symmetries:
  topological operators of dimensions $0, \cdots, d-1$, with non-invertible fusion.
  $\Rightarrow$ Formulation in terms of objects and higher-morphisms to capture the full structure

The main (surprising) point to remember is:
these are symmetries that occur in vanilla 4d Yang-Mills theories (no susy, no matter), but also very naturally realized in susy, SCFTs etc.

Non-invertible Symmetries in $d > 3$:

In the context of QFTs in $d > 3$ within the last year

[Heidenreich, McNamara, Monteiro, Reece, Rudelius, Valenzuela]

[Koide, Nagoya, Yamaguchi]

[Kaidi, Ohmori, Zheng]

[Choi, Cordova, Hsin, Lam, Shao]

[Roumpedakis, Seifnashri, Shao]

[Bhardwaj, Bottini, SSN, Tiwari]

[Choi, Cordova, Hsin, Lam, Shao]

[Kaidi, Zafrir, Zheng]

[Choi, Lam, Shao]

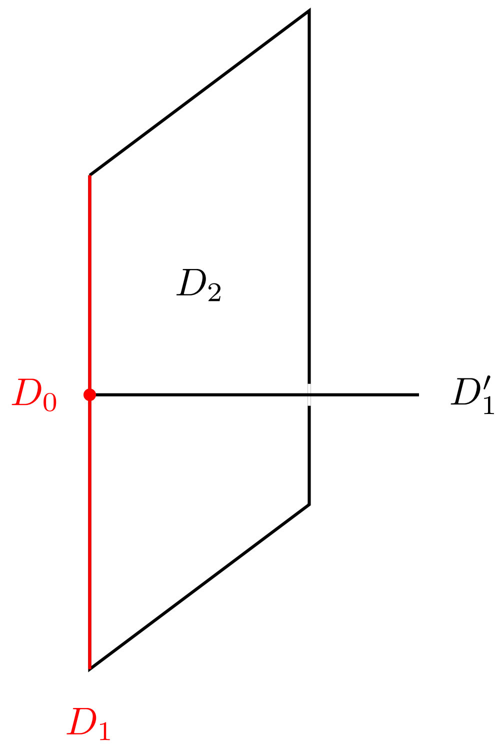
[Cordova, Ohmori]

Symmetry Categories

Consider a d-dimensional QFT $\mathfrak{T}$. Then the set of all topological defects

$$D_p^i\,, \qquad p = 0, \cdots, d-1$$

will form a $(d-1)$-category.



- Objects: D_{d-1}
- 1-morphisms D_{d-2} between objects
- 2-morphism D_{d-3} between 1-morphisms
- $\cdots$
- $(d-2)$-morphisms: local operators

Topological operators can be **genuine** or non-genuine (ends of other topological operators)

The symmetry category $\mathcal{C}_{\mathfrak{T}}$ encodes the fusion of these topological defects. "Higher fusion category"

Higher-Fusion Symmetry Cats

Clearly this structure can become quickly very unwieldy (even in topology/category theory not all the rules of this game are known)

Instead of developing the mathematical framework, I will propose a ”bottom-up”, constructive approach from QFT.

Constructive approach in QFTs

Rather than developing the general formalism we will take a constructive approach, realizing such categories in down to earth QFTs:

Distinct – and sometimes overlapping – approaches developed in the last year:

- [Kaidi, Ohmori, Zheng] Mixed anomalies to non-invertibles
- [Choi, Cordova, Hsin, Lam, Shao] Self-duality defects
- **[Bhardwaj, Bottini, SSN, Tiwari] Gauging outer automorphisms**

Examples:

- Spin$(4N)$ Yang-Mills in any dim has a $\mathbb{Z}_2^{(0)}$ outer automorphism, gauging results in Pin$^+(4N)$
- Gauging charge conjugation in Yang-Mills
- S_3-gauging of Spin(8) Yang-Mills $\Rightarrow$ allows non-abelian discrete gauging

Gauging Outer Automorphisms

The symmetry is characterized in terms of the topological defects D_p^g.

Goal: determine the topological defects and their fusion after gauging an outer automorphism.

In 3d:
was developed in condensed matter using modular tensor category tools [Barkeshli, Bonderson, M. Cheng, Z. Wang][Teo, Hughes, Fradkin].

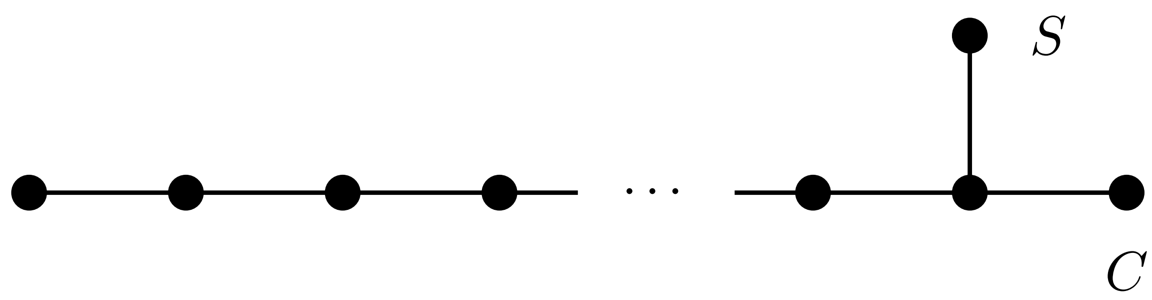
We generalize this to any dimension and propose a method to compute gauging of finite, (non-)abelian 0-form symmetries: [Bhwardwaj, Bottini, SSN, Tiwari].

Gauging Outer Automorphisms

Let $G^{(0)}$ be a finite, but not necessarily abelian, 0-form symmetry group, which acts on QFT with wlog only invertible symmetries.

Example.

Spin$(4N)$:



4d Spin$(4N)$ Yang-Mills, and the outer automorphism $G^{(0)} = \mathbb{Z}_2^{(0)}$ that exchanges the two factors of the 1-form symmetry

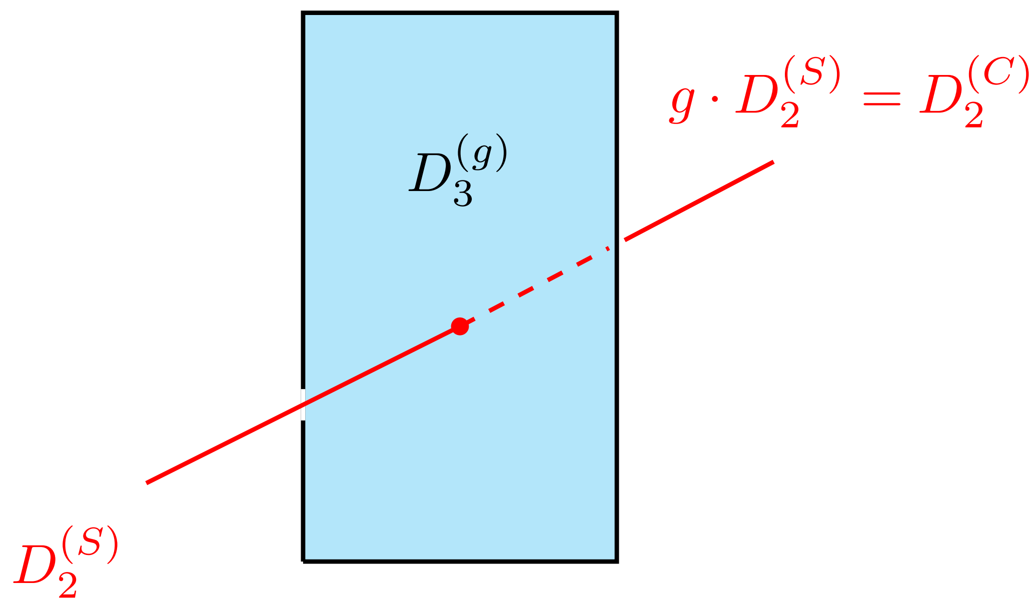
$$\Gamma^{(1)} = \mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(C)}$$

These are generated by the Gukov Witten surface operators:

$$D_2^{(S)},\ D_2^{(C)},\ D_2^{(V)} = D_2^{(S)} \otimes D_2^{(C)}, \qquad D_2^{(i)} \otimes D_2^{(i)} = D_2^{(\text{id})}$$

What is the symmetry of the theory after gauging this outer automorphism?

Action of $G^{(0)}$ on Defects



Example (cont.). Outer automorphism of Spin$(4N)$ acts on the topological defects generating the 1-form symmetry by $D_1^{(S)} \leftrightarrow D_1^{(C)}$

Gauging $\mathbb{Z}_2^{(0)}$ in 4d Spin$(4N)$ Yang-Mills gives gauge group Pin$^+(4N)$.

Non-Invertible Symmetries in 4d $\mathrm{Pin}^+(4N)$

The topological surface operators D_2^g in 4d $\mathrm{Spin}(4N)$ Yang-Mills are

$$\mathcal{C}^{\text{surfaces}}_{\mathrm{Spin}(4N)} = \left\{ D_2^{(\mathrm{id})}, D_2^{(S)}, D_2^{(C)}, D_2^{(V)} \right\}$$

There are no lines that are junctions between two distinct surface operators, but each surface has a topological line on it:

$$\mathcal{C}^{\text{lines}}_{\mathrm{Spin}(4N)} = \left\{ D_1^{(\mathrm{id})}, D_1^{(S)}, D_1^{(C)}, D_1^{(V)} \right\}$$

Also there are topological local operators $D_0^{(g)}$.
These satisfy $\mathbb{Z}_2 \times \mathbb{Z}_2$ group law

$$D_p^{(S)} \otimes D_p^{(S)} = D_p^{(\mathrm{id})}\,, \qquad D_p^{(C)} \otimes D_p^{(C)} = D_p^{(\mathrm{id})}\,, \qquad D_p^{(V)} = D_p^{(S)} \otimes D_p^{(C)}$$

The $\mathbb{Z}_2^{(0)}$ outer automorphism acts by

$$D_p^{(S)} \longleftrightarrow D_p^{(C)}\,, \qquad D_p^{(V)} \quad \text{invariant}$$

Symmetry of the 4d $\mathrm{Pin}^+(4N)$

Gauging results in the following irreducible (simple) topological **surface defects**:

$$\mathcal{C}^{\text{surfaces}}_{\mathrm{Pin}^+(4N)} = \left\{ D_2^{(\mathrm{id})}, D_2^{(SC)}, D_2^{(V)} \right\} ,$$

where

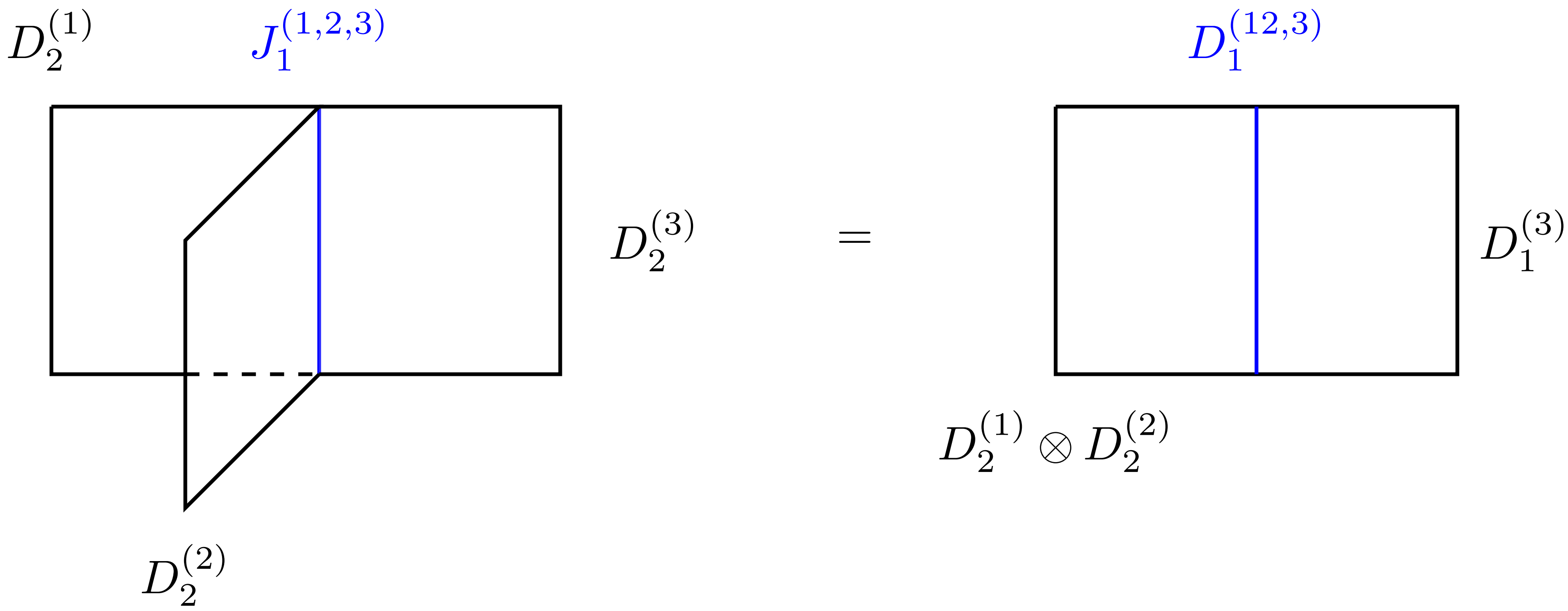
$$D_2^{(SC)} = D_2^{(S)} \oplus D_2^{(C)}$$

Naively we then find the fusion

before gauging: $\quad D_2^{(SC)} \otimes D_2^{(SC)} = 2D_2^{(\mathrm{id})} \oplus 2D_2^{(V)}$

Fusion from Junctions

To understand the fusion in the gauged theory, i.e. Pin^+ we need to determine the invariant "junctions" between the topological operators:



There are two junctions between $D_2^{(SC)} \otimes D_2^{(SC)} \to D_2^{(\mathrm{id})}$ in $\mathrm{Spin}(4N)$:

$$D_1^{(S,S;\mathrm{id})}: \; D_2^{(S)} \otimes D_2^{(S)} \to D_2^{(\mathrm{id})}$$

$$D_1^{(C,C;\mathrm{id})}: \; D_2^{(C)} \otimes D_2^{(C)} \to D_2^{(\mathrm{id})}\,.$$

Similarly, there are two junctions $D_2^{(SC)} \otimes D_2^{(SC)} \to D_2^{(V)}$

$$
\begin{aligned}
D_1^{(S,C;V)} &: \; D_2^{(S)} \otimes D_2^{(C)} \to D_2^{(V)} \\
D_1^{(C,S;V)} &: \; D_2^{(C)} \otimes D_2^{(S)} \to D_2^{(V)} \,.
\end{aligned}
$$

These 1-morphisms are exchanged by the gauging:

$$
\begin{aligned}
D_1^{(S,S;\mathrm{id})} &\longleftrightarrow D_1^{(C,C;\mathrm{id})} \\
D_1^{(C,S;V)} &\longleftrightarrow D_1^{(S,C;V)} \,.
\end{aligned}
$$

Thus, we have single map

$$
\begin{aligned}
D_1^{(SC,SC;\mathrm{id})} &: \; D_2^{(SC)} \otimes D_2^{(SC)} \to D_2^{(\mathrm{id})} \\
D_1^{(SC,SC;V)} &: \; D_2^{(SC)} \otimes D_2^{(SC)} \to D_2^{(V)}
\end{aligned}
$$

in $\mathrm{Pin}^+(4N)$, and hence we obtain the fusion rule

$$
D_2^{(SC)} \otimes D_2^{(SC)} = D_2^{(\mathrm{id})} \oplus D_2^{(V)}
$$

Symmetries of Pin^+: Topological lines

Gauging a 0-form symmetry (in any dimensions) gives a dual $\Gamma^{(d-2)}$ generated by topological lines, so we gain a $\mathbb{Z}_2$ topological line

$$D_1^{(-)}\,, \qquad D_1^{(-)} \otimes D_1^{(-)} = D_1^{(\mathrm{id})}$$

On $D_2^{(V)}$ we can have the trivial line $D_1^{(V)}$, but also a new line, $D_1^{(V)} \otimes D_1^{(-)} = D_1^{(V-)}$.

The full set of topological lines is:

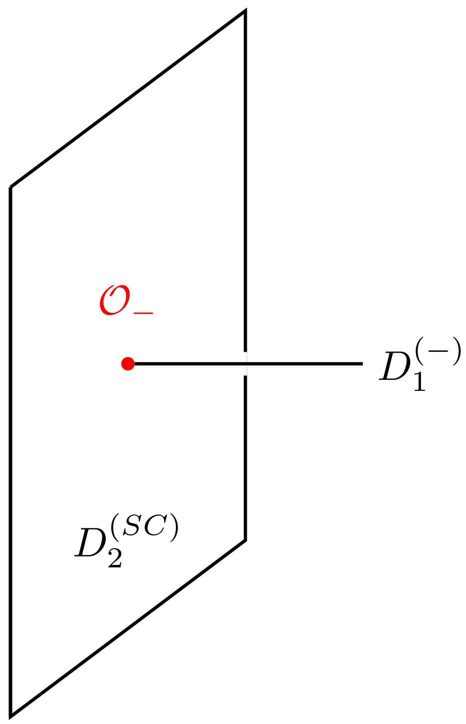
$$\mathcal{C}^{\text{lines}}_{\text{Pin}^+(4N)} = \left\{ D_1^{(\mathrm{id})}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V_-)} \right\}$$

Symmetries of Pin^+: Topological point-operators

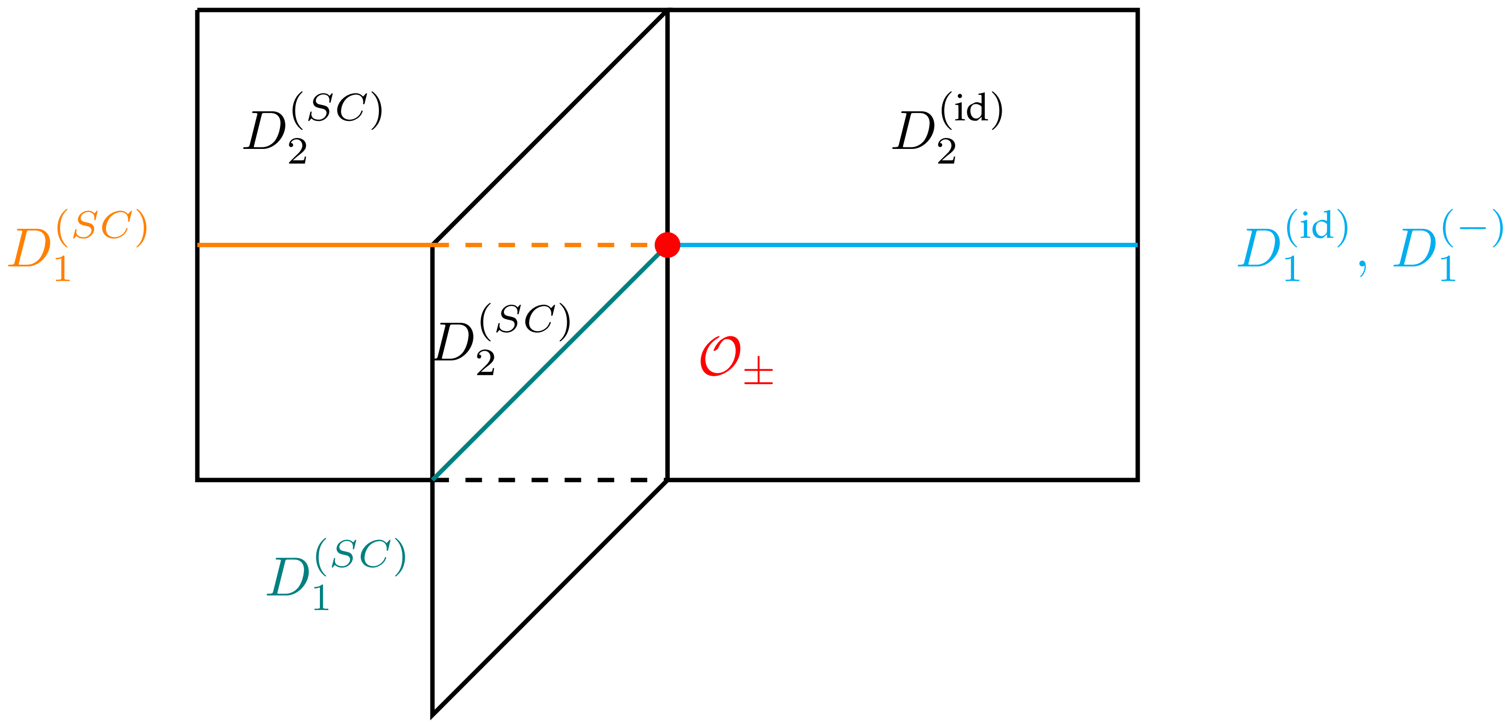
On $D_2^{(S)} \oplus D_2^{(C)}$ there were two local operators before gauging $D_0^{(S)}$ and $D_0^{(C)}$. After gauging: they decompose into irreps of $\mathbb{Z}_2$

$$\mathcal{O}_\pm = D_0^{(S)} \pm D_0^{(C)}$$

$\mathcal{O}_+$ is invariant and genuine local operator on $D_2^{(SC)}$, but $\mathcal{O}_-$ is charged under $\mathbb{Z}_2$, and after gauging is not a genuine operator, i.e. we need to attach the line $D_1^{(-)}$:



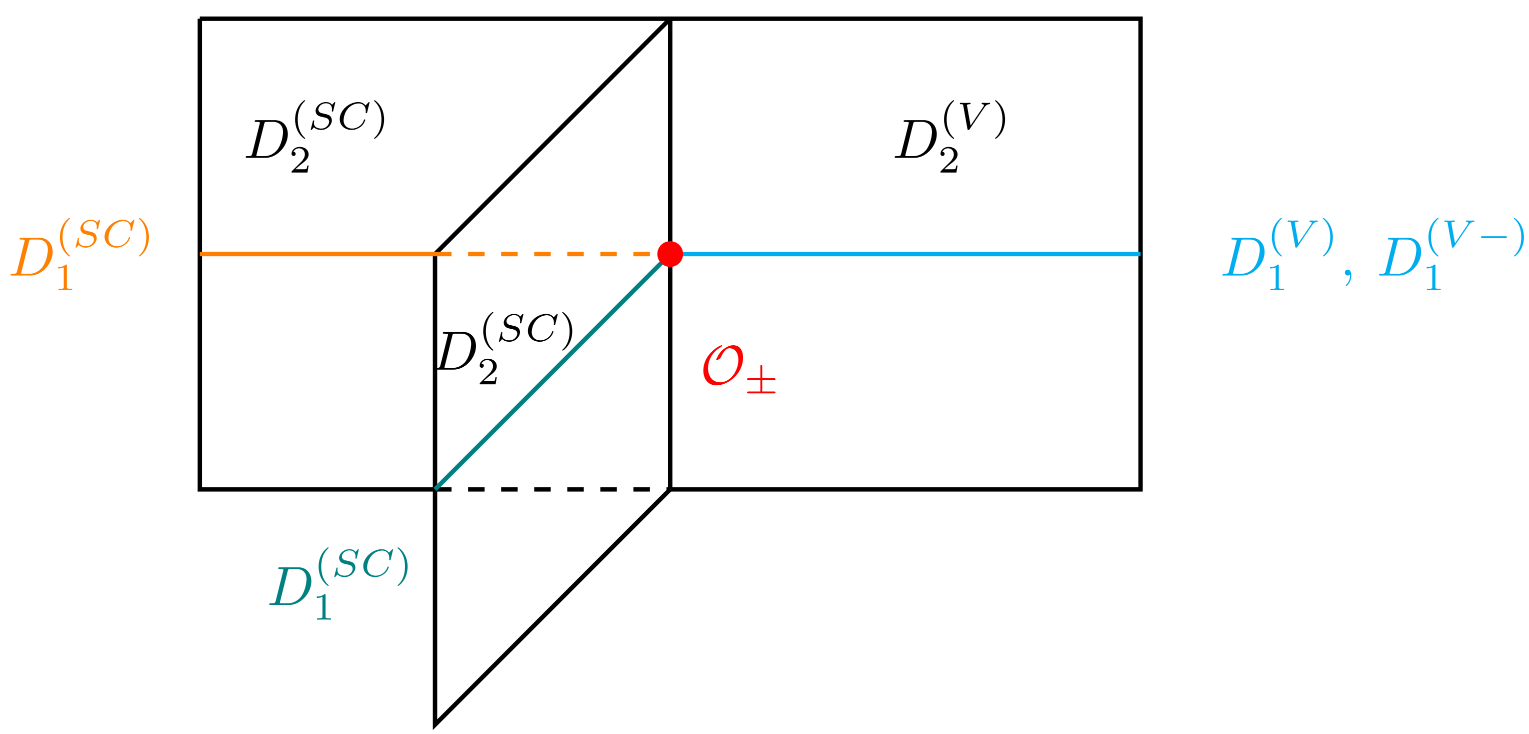
Interpretation of this? Higher-fusion:



Thus, there is a non-invertible fusion on the lines:

$$D_1^{(SC)} \otimes D_1^{(SC)} \supset D_1^{(\mathrm{id})} \oplus D_1^{(-)}$$

Similarly $D_2^{(SC)} \otimes D_2^{(SC)}$ can fuse also into $D_2^{(V)}$:



Thus, there is a non-invertible fusion on the lines:

$$D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(\mathrm{id})} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V_-)}$$

Global Fusion

So far we have discussed local fusion: i.e. the fusion without taking into account global aspects of the spacetime manifold M_d.

In a nutshell: the global fusion of p-dimensional topological defects $D_p^{(1)} \otimes D_p^{(2)}$ is obtained by gauging the symmetry localized on the defect $D_p^{(1)} \otimes D_p^{(2)}$.

E.g. for 4d $\mathrm{Pin}^+(4N)$ the fusion of surfaces

$$D_2^{(SC)} \otimes D_2^{(SC)} = D_2^{(\mathrm{id})} \oplus D_2^{(V)}$$

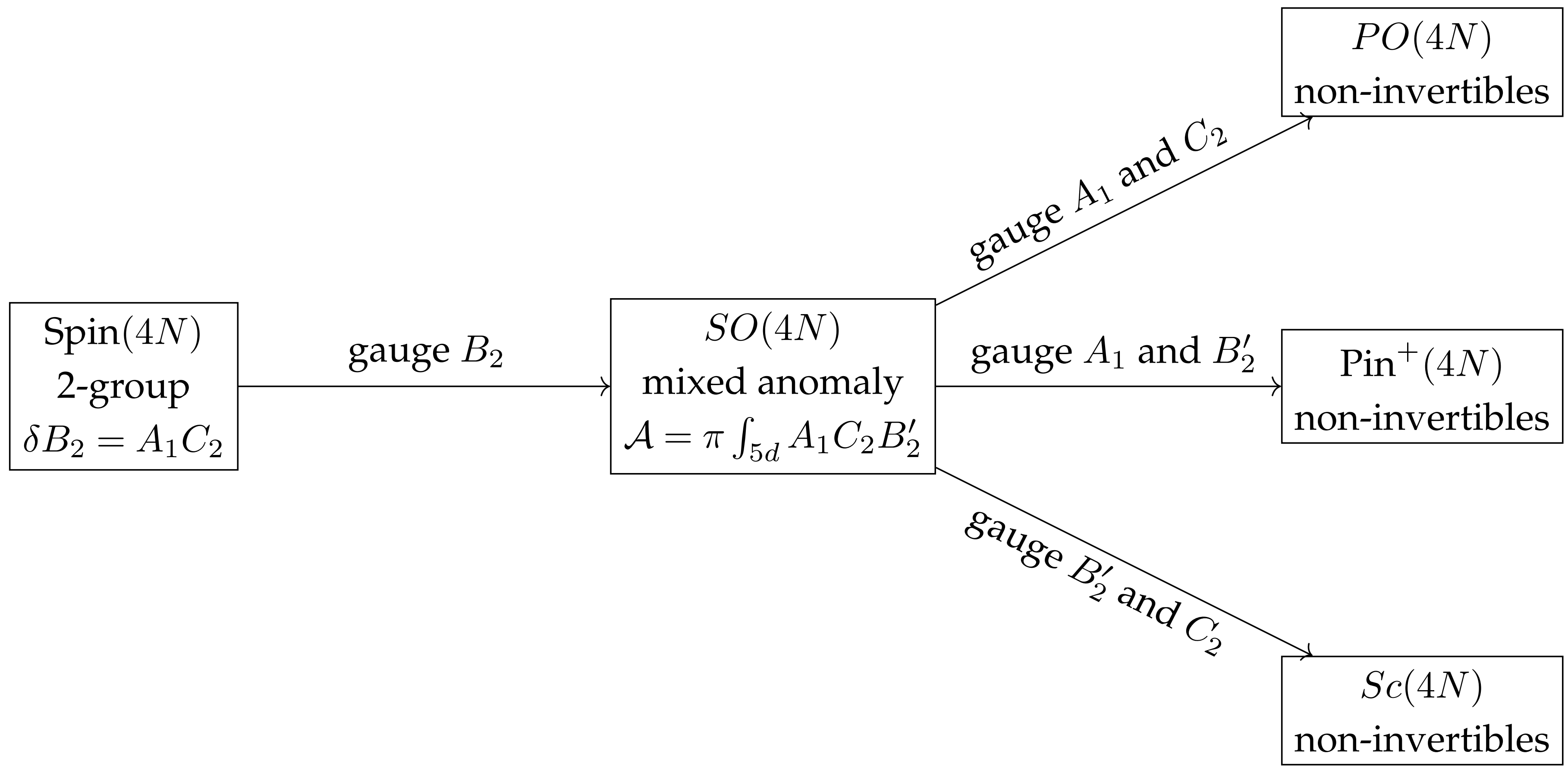
becomes on a 2-manifold M_2

$$D_2^{(SC)}(M_2) \otimes D_2^{(SC)}(M_2) = \frac{D_2^{(\mathrm{id})}}{\mathbb{Z}_2}(M_2) \oplus \frac{D_2^{(V)}}{\mathbb{Z}_2}(M_2)$$

where $\frac{D_2^{(i)}}{\mathbb{Z}_2}(M_2)$ for $i \in \{\mathrm{id}, V\}$ denotes the surface defect obtained by gauging the $\mathbb{Z}_2$ 0-form symmetry of $D_2^{(i)}$ wrapped along M_2. These are precisely condensation defects – see [Gaiotto, Johnson-Freyd][Choi, Cordova, Hsin, Lam Shao][Rumpedakis, Seifnashri, Shao]

For abelian G we can compare to the complementary approach by [Kaidi, Ohmori, Zheng] (and a slightly different approach [Choi, Cordova, Lam, Shao]), who derived non-invertible symmetries by gauging mixed anomalies.

Indeed, the $\mathrm{Pin}^+(4N)$ non-invertible symmetries can be obtained also by following the KOZ approach:



Non-Invertible in 5d

5d $\mathcal{N} = 2$ Spin$(4N)$ super Yang-Mills has a 3-categorical symmetry

$$\left\{ D_i^{(\text{id})}, D_i^{(S)}, D_i^{(C)}, D_i^{(V)} \right\}, \qquad i = 3, 2, 1$$

Gauging $\mathbb{Z}_2^{(0)}$ outer automorphism we get simple objects

$$\mathcal{C}^i_{\text{Pin}^+(4N)} = \left\{ D_i^{(\text{id})}, D_i^{(SC)}, D_i^{(V)} \right\}$$

$i = 3$: objects; $i = 2$: 1-endomorphisms. These have fusion

$$D_i^{(SC)} \otimes D_i^{(SC)} = D_i^{(\text{id})} \oplus D_i^{(V)}$$

The 2-endomorphisms are

$$\mathcal{C}^{\text{2-endo}}_{\text{Pin}^+(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V_-)} \right\}$$

with fusion of TY type

$$D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(\text{id})} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V_-)}$$

The non-trivial global fusion is

$$D_3^{(SC)}(\Sigma_3) \otimes D_3^{(SC)}(\Sigma_3) = \frac{D_3^{(\text{id})}}{\mathbb{Z}_2^{(1)}}(\Sigma_3) \oplus \frac{D_3^{(V)}}{\mathbb{Z}_2^{(1)}}(\Sigma_3)$$

where

$$\begin{aligned} \frac{D_3^{(\text{id})}}{\mathbb{Z}_2^{(1)}} &: \quad \mathbb{Z}_2^{(1)} \quad \text{generated by } D_1^{(\text{id})}, D_1^{(-)} \\ \frac{D_3^{(V)}}{\mathbb{Z}_2^{(1)}} &: \quad \mathbb{Z}_2^{(1)} \quad \text{generated by } D_1^{(V)}, D_1^{(V_-)} \end{aligned}$$

Scope and Extensions

Huge scope to construct non-invertible symmetries, which you can find in [Bhardwaj, Bottini, SSN, Tiwari]:

1. Non-invertible symmetries for disconnected gauge groups $O(2)$ (consistent with [Heidenreich, McNamara, Monteiro, Reece, Rudelius, Valenzuela])and $\widetilde{SU}(N)$ (gauging charge conjugation)
2. Allows gauging non-abelian finite symmetries
3. Non-invertible 3-categorical symmetries in absolute 6d $(2,0)$ theories, e.g.

$$\left[SO(2n) \times SO(2n)\right] \rtimes \mathbb{Z}_2$$

4. Non-invertible symmetries in 5d SCFTs

Non-Invertibles and Deconfining/Confining Vacua

Recall, 1-form symmetries provide order parameters for confinement: $\Gamma^{(1)}$ unbroken/broken corresponds to confining/deconfining vacuum.

Non-invertibles arise in 4d $\mathcal{N} = 1$ SYM from mixed 't Hooft anomaly between chiral symmetry and 1-form symmetry and provide non-invertible domain walls between confining and deconfining vacua – depending on the global form of the gauge group [Bottini, SSN; wip].

Example $\mathfrak{su}(2)$: Line operators are generated by W Wilson and H 't Hooft lines. Unbroken $\Gamma^{(1)}$ in the two vacua (monopole and dyon condensing vacuum) for each polarization (aka global form of the gauge group):

Genuine line operators	G	$\Gamma^{(1)}_m$	$\Gamma^{(1)}_d$
$< W >$	$SU(2)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$< H >$	$SO(3)_+$	$\emptyset$	$\mathbb{Z}_2$
$< H + W >$	$SO(3)_-$	$\mathbb{Z}_2$	$\emptyset$

Physical Implications of non-Invertible Symmetries

We know that between the two confining vacua of $SU(2)$ there is an invertible domain wall. What about $SO(3)$?

There is a mixed 't Hooft anomaly $\mathbb{Z}_4^{(0)}$ and $\mathbb{Z}_2^{(1)}$ with 't Hooft anomaly

$$\mathcal{A} = \pi \int_{M_5} A_1 \cup \frac{\mathfrak{P}(B_2)}{2}$$

Let $D_3^{(n)}$ generate $\mathbb{Z}_4^{(0)}$, with $D_3^{(2)}$ generating the the non-anomalous $\mathbb{Z}_2^{(0)}$ subgroup.

Apply the [Kaidi, Ohmori, Zheng] mixed anomaly approach to this setup: To gauge the 1-form symmetry, the defect $D_3^{(1)}$ needs to be dressed by a TQFT (as it is not gauge invariant due to the mixed anomaly)

$$\mathcal{N}_3^{(1)} = D_3^{(1)} \times \mathcal{T}$$

where $\mathcal{T}$ is the minimal TQFT with anomaly $\mathfrak{P}(B)$ [Hsin, Lam, Seiberg]: $U(1)_2$ CS.

Physical Implications of non-Invertible Symmetries

In $SO(3)$ SYM we then have a non-invertible defect

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)} = \frac{D_3^{(2)}}{|H^0(M_3,\mathbb{Z}_2)|} \left(\sum_{M_2 \in H_2(M_3,\mathbb{Z}_2)} (-1)^{\chi(M_2)} D_1(M_2) \right)$$

$$\mathcal{N}_3^{(1)} \otimes \overline{\mathcal{N}}_3^{(1)} = \frac{1}{|H^0(M_3,\mathbb{Z}_2)|} \left(\sum_{M_2 \in H_2(M_3,\mathbb{Z}_2)} (-1)^{\chi(M_2)} D_1(M_2) \right)$$

where $D_1(M_2) = e^{i\pi \oint_{M_2} B_2}$ generates $\Gamma^{(1)}$ of $SO(3)$.

In $SO(3)$ this non-invertible defect corresponds to the domain wall between the confining and deconfining vacua.

Can be applied in all "reading between the lines" [Aharony, Tachikawa, Seiberg] global forms, to study the IR behavior of the $\mathcal{N}=1$ SYM theories. [Bottini, SSN; wip]

Outlook

Generalized symmetries – higher form, higher group and non-invertible symmetries – are ubiquitous in QFTs

1. Learn to gauge higher-form/non-invertible symmetries in higher-categories; 't Hooft anomalies
2. Physical implications of these symmetries (confinement, pion decay etc)
3. Develop a mathematically sound framework for higher fusion categories (higher meaning ≥ 2)
4. Is this the most general "symmetry structure" for QFTs?
5. String theory realization of non-invertibles (e.g. using SymTFT)

Thank you!